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EQUIVALENCE OF $\mathbf{u}$–p AND ζ–ψ FORMULATIONS OF THE TIME-DEPENDENT NAVIER–STOKES EQUATIONS

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SUMMARY

This paper deals with the non-stationary incompressible Navier–Stokes equations for two-dimensional flows expressed in terms of the velocity and pressure and of the vorticity and streamfunction. The equivalence of the two formulations is demonstrated, both formally and rigorously, by virtue of a condition of compatibility between the boundary and initial values of the normal component of velocity. This condition is shown to be the only compatibility condition necessary to allow for solutions of a minimal regularity, namely H^1 for the velocity, as in most current numerical schemes relying on spatial discretizations of local type.

KEY WORDS Incompressible flows Navier–Stokes equations Primitive variables Vorticity–Streamfunction Compatibility conditions

1. INTRODUCTION

The formulation of the incompressible Navier–Stokes equations in terms of the so-called non-primitive variables vorticity and streamfunction represents the most popular approach for the study of steady and unsteady viscous flows in two dimensions. The equivalence of the vorticity–streamfunction equations with the original primitive variable formulation of the viscous incompressible problem is well established only for the steady state equations, which constitute a standard elliptic boundary value problem. In the time-dependent case the governing equations constitute instead a mixed initial–boundary value problem and to our knowledge a proof of the equivalence of the vorticity–streamfunction and velocity–pressure formulations in the presence of solid boundaries does not seem to be available to the fluid dynamics community.

The present paper intends to show that the equivalence of these two formulations of the non-stationary Navier–Stokes equations can be demonstrated, both formally and rigorously, provided that the normal component of the boundary value of the velocity is compatible with that of the initial velocity. Such a compatibility condition together with the solenoidality condition for the initial velocity field allow for an optimal choice of the linear space to which the initial datum should belong. The optimal setting in question has been provided by

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Ladyzhenskaya (Reference 1, p. 88) and Temam (Reference 2, p. 253) and includes the optimal condition of compatibility between the data specified for the (normal component of) velocity on the boundary and at the initial time, such that the existence and uniqueness of the solution are an easy consequence of Lions' theorem (Reference 3, p. 257; Reference 4, p. 218).

It should be remarked that the same conditions of solenoidality and compatibility between the boundary and initial data are also necessary to prove the existence and uniqueness of classical solutions of the time-dependent 2D Euler equations for an incompressible ideal fluid of zero viscosity.[5] Thus the presence of the compatibility condition is due only to the incompressibility, irrespective of the viscous or inviscid character of the fluid.

Additional compatibility conditions concerning the tangential components of the initial velocity field and the boundary condition have been considered for the viscous equations. However, the condition on the tangential components of the initial velocity is unnecessarily stringent. As shown further, the solenoidality of the initial velocity and the compatibility of the normal component of the boundary value of the velocity with that of the initial velocity field are all that is needed for ensuring existence of a solution with some *minimal* regularity, whereas compatibility of the tangential components of the initial and boundary data is required only if higher regularity is desired (see e.g. Reference 6).

It can also be noted that there are computational fluid dynamicists who believe that no compatibility condition exists between the initial and boundary data for the incompressible Navier–Stokes problem, exactly like none exists for the parabolic equation governing the diffusion of temperature in a heat-conducting medium. After all, the equations governing the motion of a viscous fluid define a parabolic problem, so that no basic difference is expected in the mathematical structure with respect to the diffusion equation. However, this argument is not completely correct, because it neglects the role played by the incompressibility in the mathematical theory of the Navier–Stokes equations. In fact, the argument denying the existence of any compatibility condition between initial and boundary data is correct only as far as the tangential components of the velocity are concerned, while it is false when referred to the component normal to the boundary and to a vector field which must be solenoidal. This misunderstanding can explain why the importance of the aforementioned compatibility condition in viscous incompressible flows has not been fully recognized so far in the CFD community. In this connection it may be worthwhile to remark that the unsteady incompressible Navier–Stokes equations do define a parabolic problem, but only *after* it has been projected on to the space of solenoidal vector fields tangential to the boundary, and this means taking an initial velocity field which satisfies the compatibility condition.

The content of the paper is organized as follows. In Section 2 the complete statement of the unsteady Navier–Stokes equations governing the primitive variables velocity and pressure is given. Section 3 introduces the non-primitive variable representation of the equations in terms of vorticity and streamfunction for two-dimensional flows. This section also provides a formal proof of the equivalence of the two formulations for unsteady flows, whose understanding does not require tools of functional analysis. Section 4 contains the definitions and preliminaries which provide the mathematical framework needed for rigorous treatment of the problem. In Section 5 classical results concerning the velocity–pressure formulation are recalled so as to be able to show the well-posedness of the time-dependent problem expressed in terms of vorticity and streamfunction variables. Section 6 finally demonstrates the equivalence of the two considered representations of unsteady incompressible viscous flows in two dimensions.

2. INCOMPRESSIBLE NAVIER–STOKES EQUATIONS

The motion of a viscous incompressible fluid is governed by the Navier–Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f}, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

where $\mathbf{u}(\mathbf{x}, t)$ is the velocity, $p(\mathbf{x}, t)$ is the pressure and ν is the (constant) coefficient of kinematic viscosity, the constant density of the fluid having been absorbed into the pressure. The term $\mathbf{f}(\mathbf{x}, t)$ on the right-hand side of the momentum equation (1) represents the body forces.

The statement of the problem is made complete by the specification of suitable boundary and initial conditions. A typical boundary condition consists of prescribing the value of the velocity $\mathbf{b}$ on the boundary,

$$\mathbf{u}|_{\partial\Omega} = \mathbf{b}, \tag{3}$$

where $\partial\Omega$ is the boundary of the domain Ω occupied by the fluid and $\mathbf{b} = \mathbf{b}(\mathbf{x}_{\partial\Omega}, t)$. When the boundary is a solid wall in contact with the fluid, the velocity boundary value $\mathbf{b}$ is equal to the velocity of the wall. In this case the boundary condition for the tangential velocity is usually referred to as a *no-slip* condition. In the following we restrict our analysis to 2D problems and assume that the fluid domain Ω is bounded and simply connected, which means that it is of finite extent and contains no holes. Furthermore, Ω is assumed to be open and $\partial\Omega$ to be smooth enough, say Lipschitz-continuous.

The initial condition consists of the specification of the velocity field $\mathbf{u}_0$ at the initial time $t = 0$, namely,

$$\mathbf{u}|_{t=0} = \mathbf{u}_0(\mathbf{x}). \tag{4}$$

The boundary velocity $\mathbf{b}$ must satisfy for all $t \geqslant 0$ the global condition

$$\oint_{\partial\Omega} \mathbf{n} \cdot \mathbf{b} \, d\Gamma = 0, \tag{5}$$

which follows from integrating the continuity equation $\nabla \cdot \mathbf{u} = 0$ over Ω. On the other hand, the initial velocity field $\mathbf{u}_0$ is assumed to be solenoidal, i.e.

$$\nabla \cdot \mathbf{u}_0 = 0. \tag{6}$$

Finally, the boundary and initial data $\mathbf{b}$ and $\mathbf{u}_0$ are assumed to satisfy the compatibility condition

$$\mathbf{n} \cdot \mathbf{b}|_{t=0} = \mathbf{n} \cdot \mathbf{u}_0|_{\partial\Omega}. \tag{7}$$

The subsequent analysis will show that the condition (6) on the initial data and the compatibility condition (7) between the initial and boundary data are required to establish the equivalence of the present velocity–pressure formulation and the vorticity–streamfunction formulation. As a matter of fact, they are also necessary to prove the existence and uniqueness of classical solutions of the non-stationary Euler equations in two dimensions for an inviscid incompressible fluid.[5]

The compatibility condition (7) is *not* satisfied for problems characterized by an impulsive initial motion of bodies or walls in contact with the fluid. However, in these situations the proper initial condition for the flow around the body is given by a potential flow that establishes itself, by virtue of the incompressibility, in response to the sudden motion of the boundary; see e.g. the discussions in Reference 7 (p. 80) and Reference 8 (p. 81). The initial potential velocity field is caused by the 'jump' between the different values prescribed on the normal velocity by the boundary condition and by the initial condition. More precisely, if $\mathbf{n}\cdot\mathbf{b}|_{t=0} \neq \mathbf{n}\cdot\mathbf{u}_0|_{\partial\Omega}$, one introduces a velocity potential Φ_0 solution to the Neumann problem

$$-\nabla^2\Phi_0 = 0, \qquad \mathbf{n}\cdot\nabla\Phi_0|_{\partial\Omega} = \mathbf{n}\cdot\mathbf{b}|_{t=0} - \mathbf{n}\cdot\mathbf{u}_0|_{\partial\Omega}, \tag{8}$$

whose solvability condition $\oint \mathbf{n}\cdot(\mathbf{b}|_{t=0} - \mathbf{u}_0|_{\partial\Omega})\,d\Gamma = 0$ is satisfied by conditions (5) and (6). Then the initial velocity is replaced by a 'modified' initial velocity $\mathbf{u}_0^*$ defined as

$$\mathbf{u}_0^* = \mathbf{u}_0 + \nabla\Phi_0. \tag{9}$$

With this modified initial field the compatibility condition (7) is automatically satisfied, since

$$\begin{aligned}\mathbf{n}\cdot\mathbf{u}_0^*|_{\partial\Omega} &= \mathbf{n}\cdot(\mathbf{u}_0 + \nabla\Phi_0)|_{\partial\Omega}\\ &= \mathbf{n}\cdot\mathbf{u}_0|_{\partial\Omega} + \mathbf{n}\cdot\mathbf{b}|_{t=0} - \mathbf{n}\cdot\mathbf{u}_0|_{\partial\Omega}\\ &= \mathbf{n}\cdot\mathbf{b}|_{t=0}\end{aligned}$$

by virtue of the boundary condition imposed on Φ_0 in problem (8). Thus, provided that the initial velocity field is modified according to (9), the fulfilment of the compatibility condition (7) between the boundary and initial data can be ensured, even for problems involving an impulsive motion of the boundaries. It should be noted that in these cases a discontinuity in the tangential components of the velocity on the boundary is usually produced by the introduction of the initial potential flow $\nabla\Phi_0$, namely it results in general that

$$\mathbf{n}\times\mathbf{b}|_{t=0} \neq \mathbf{n}\times\mathbf{u}_0^*|_{\partial\Omega}. \tag{10}$$

This means that in the case of an impulsive start the H^1-norm of the velocity field necessarily blows up as $t \to 0$.

Note that this loss of regularity of the solution is more dramatic than that pointed out by Heywood and Rannacher (Reference 6, p. 277). Indeed, they assumed no-slip conditions at all times and observed a loss of regularity in the H^3-norm of the velocity field as $t \to 0$ unless the data satisfy some (non-local and virtually uncheckable) compatibility conditions that enable the initial pressure to satisfy an overdetermined Poisson problem.

3. VORTICITY–STREAMFUNCTION EQUATIONS

Coming to the numerical solution of the Navier–Stokes equations, a serious difficulty is met in the determination of the pressure field and in the fulfilment of the incompressibility condition. In fact, the continuity equation (2) is somewhat peculiar in that it represents a *constraint* for the velocity field. At the same time the pressure variable, which appears in the momentum equation through the term ∇p, provides the degrees of freedom necessary to accommodate and satisfy such a constraint. Correspondingly, no dynamical equation exists for the pressure, so that in incompressible problems this variable does not have the usual thermodynamical meaning. Here the role of the pressure is that of adjusting itself instantaneously in order that the condition of zero divergence be satisfied at every time. This behaviour is related to the well-known fact that in an incompressible fluid the value of the speed of sound becomes infinite. As a consequence, the pressure field cannot be calculated by an explicit time advancement procedure but requires instead an implicit determination capable of taking into account the coupling existing between the pressure and the velocity as well as the effect of the velocity boundary condition. This aspect can be considered the most distinctive feature of the primitive variable formulation of the incompressible Navier–Stokes equations.

A well-known method for circumventing this kind of difficulty in the solution of two-dimensional problems consists of eliminating the pressure variable altogether and introducing two scalar functions, the vorticity and the streamfunction, as unknown variables. For fluid motions parallel to the plane xy the vorticity ζ is the z-component of the vorticity vector $\boldsymbol{\zeta} = \nabla \times \mathbf{u}$ normal to that plane, namely

$$\zeta = \nabla \times \mathbf{u} \cdot \mathbf{k} \quad \text{or} \quad \zeta\mathbf{k} = \nabla \times \mathbf{u}, \tag{11}$$

where $\mathbf{u} = (u_x, u_y)$, $\nabla = (\partial/\partial x, \partial/\partial y)$ and $\mathbf{k}$ is the unit vector normal to the plane xy. In two dimensions the condition of incompressibility $\nabla \cdot \mathbf{u} = 0$ can be satisfied exactly by expressing $\mathbf{u}$ in terms of a streamfunction ψ according to

$$\mathbf{u} = \nabla\psi \times \mathbf{k}. \tag{12}$$

This equation, once expressed in terms of the vector components, gives $u_x = \partial\psi/\partial y$ and $u_y = -\partial\psi/\partial x$. Thus one obtains immediately

$$\nabla \cdot \mathbf{u} = \nabla \cdot (\nabla\psi \times \mathbf{k}) = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial\psi}{\partial x}\right) = 0.$$

We now eliminate the pressure from the Navier–Stokes equations by taking the curl of the momentum equation. To simplify the derivation, the non-linear term is first expressed in the so-called Lamb form, namely

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = (\nabla \times \mathbf{u}) \times \mathbf{u} + \nabla(\tfrac{1}{2}u^2) = \zeta\mathbf{k} \times \mathbf{u} + \nabla(\tfrac{1}{2}u^2).$$

Then the application of the curl operator to the non-linear term gives

$$\nabla \times [(\mathbf{u} \cdot \nabla)\mathbf{u}] = \nabla \times [\zeta\mathbf{k} \times \mathbf{u} + \nabla(\tfrac{1}{2}u^2)] = \nabla \times [\zeta\mathbf{k} \times \mathbf{u}],$$

so that the curl of the momentum equation (1) gives the equation

$$\frac{\partial}{\partial t}(\zeta\mathbf{k}) + \nabla \times (\zeta\mathbf{k} \times \mathbf{u}) = \nu\nabla^2(\zeta\mathbf{k}) + g\mathbf{k},$$

where $g = \nabla \times \mathbf{f} \cdot \mathbf{k}$. Consider now the vector identity

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}\nabla \cdot \mathbf{b} - \mathbf{b}\nabla \cdot \mathbf{a} + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$$

and use it with $\mathbf{a} = \nabla \times \mathbf{u}$, $\mathbf{b} = \mathbf{u}$ and $\nabla \cdot \mathbf{u} = 0$ to give

$$\nabla \times [(\nabla \times \mathbf{u}) \times \mathbf{u}] = (\mathbf{u} \cdot \nabla)\nabla \times \mathbf{u} - (\nabla \times \mathbf{u} \cdot \nabla)\mathbf{u}.$$

In the present 2D situation $\nabla \times \mathbf{u} = \zeta\mathbf{k}$, while the second term on the right-hand side vanishes, since $\mathbf{u}$ does not depend on z. It follows that $\nabla \times [\zeta\mathbf{k} \times \mathbf{u}] = (\mathbf{u} \cdot \nabla)\zeta\mathbf{k}$, so that the curl of the non-linear term in two dimensions can be expressed in the form

$$\nabla \times [(\mathbf{u} \cdot \nabla)\mathbf{u}] = \nabla \times (\zeta\mathbf{k} \times \mathbf{u}) = (\mathbf{u} \cdot \nabla)\zeta\mathbf{k}.$$

By virtue of the representation $\mathbf{u} = \nabla\psi \times \mathbf{k}$, the curl of the non-linear term can be given the final form

$$\nabla \times [(\mathbf{u} \cdot \nabla)\mathbf{u}] = \frac{\partial(\zeta, \psi)}{\partial(x, y)}\mathbf{k} = J(\zeta, \psi)\mathbf{k},$$

where, as usual, J denotes the Jacobian determinant.

In conclusion, taking the curl of the momentum equation leads to the vorticity transport equation

$$\frac{\partial\zeta}{\partial t} + J(\zeta, \psi) = \nu\nabla^2\zeta + g.$$

On the other hand, substituting the expression $\mathbf{u} = \nabla\psi \times \mathbf{k}$ into the vorticity definition (11) gives the following Poisson equation for the streamfunction:

$$-\nabla^2\psi = \zeta.$$

The boundary conditions supplementing the two equations above are deduced by separating the normal and tangential components of the velocity boundary condition $\mathbf{u}|_{\partial\Omega} = \mathbf{b}$. Here $\partial\Omega$ represents the boundary of the two-dimensional domain Ω, which is always assumed to be simply connected. Let $\mathbf{n}$ denote the outward unit vector normal to the boundary $\partial\Omega$ and $\boldsymbol{\tau}$ the unit vector tangential to $\partial\Omega$ with anticlockwise orientation. Finally, let s be the curvilinear co-ordinate along the boundary $\partial\Omega$. Then the boundary condition $\mathbf{u}|_{\partial\Omega} = \mathbf{b}$ yields the condition for the normal component,

$$\mathbf{n} \cdot \nabla\psi \times \mathbf{k}|_{\partial\Omega} = \mathbf{k} \times \mathbf{n} \cdot \nabla\psi|_{\partial\Omega} = \boldsymbol{\tau} \cdot \nabla\psi|_{\partial\Omega} = \left.\frac{\partial\psi}{\partial s}\right|_{\partial\Omega} = \mathbf{n} \cdot \mathbf{b},$$

and for the tangential component,

$$\boldsymbol{\tau} \cdot \nabla\psi \times \mathbf{k}|_{\partial\Omega} = \mathbf{k} \times \boldsymbol{\tau} \cdot \nabla\psi|_{\partial\Omega} = -\mathbf{n} \cdot \nabla\psi|_{\partial\Omega} = -\left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega} = \boldsymbol{\tau} \cdot \mathbf{b}.$$

The first boundary condition, after integrating its right-hand side, provides a Dirichlet condition for ψ. By virtue of the global condition $\oint_{\partial\Omega} \mathbf{n} \cdot \mathbf{b}\, \mathrm{d}s = 0$, such an integration defines a single-valued function up to an arbitrary additive function of time, namely

$$a(s, t) = \int_{s_1}^{s} \mathbf{n}(s') \cdot \mathbf{b}(s', t)\, \mathrm{d}s' + A(t),$$

where s_1 is the co-ordinate of any fixed point of $\partial\Omega$ and s' is a dummy variable of integration. To simplify the expression of the boundary conditions for ψ, we drop the term $A(t)$ from the Dirichlet condition and introduce the notation

$$b(s, t) = -\boldsymbol{\tau}(s) \cdot \mathbf{b}(s, t),$$

so that the two conditions can be written as

$$\psi|_{\partial\Omega} = a, \qquad \left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega} = b.$$

(Note that b should not be confused with $|\mathbf{b}|$, i.e. $b \neq |\mathbf{b}|$.)

As far as the initial condition for the system of equations governing ζ and ψ is concerned, the initial velocity field $\mathbf{u}_0$ provides the following initial condition for the vorticity:

$$\zeta|_{t=0} = \nabla \times (\mathbf{u}|_{t=0}) \cdot \mathbf{k} = \nabla \times \mathbf{u}_0 \cdot \mathbf{k}.$$

Collecting the equations and conditions all together, the vorticity–streamfunction formulation of the Navier–Stokes problem for two-dimensional flows is

$$\begin{cases} \dfrac{\partial \zeta}{\partial t} - \nu \nabla^2 \zeta + J(\zeta, \psi) = g, \\ -\nabla^2 \psi = \zeta, \\ \psi|_{\partial\Omega} = a, \qquad \left.\dfrac{\partial \psi}{\partial n}\right|_{\partial\Omega} = b, \\ \zeta|_{t=0} = \nabla \times \mathbf{u}_0 \cdot \mathbf{k}, \end{cases} \tag{13}$$

where $g = \nabla \times \mathbf{f} \cdot \mathbf{k}$, $a = \int_{s_1}^{s} \mathbf{n} \cdot \mathbf{b} \, ds'$ and $b = -\boldsymbol{\tau} \cdot \mathbf{b}$. The initial datum $\mathbf{u}_0(\mathbf{x})$ and the boundary datum $a(s, t)$ are assumed to satisfy the conditions

$$\nabla \cdot \mathbf{u}_0 = 0, \qquad \frac{\partial a(s, 0)}{\partial s} = \mathbf{n} \cdot \mathbf{u}_0|_{\partial\Omega}, \tag{14}$$

the latter being nothing but the compatibility condition (7) rewritten in terms of the Dirichlet datum $a = a(s, t)$. The global condition $\oint_{\partial\Omega} \mathbf{n} \cdot \mathbf{b} \, ds = 0$ does not appear any more because it has already been taken into account in the definition of the single-valued function $a(s, t)$.

Theorem 1

The vorticity–streamfunction problem (13) is equivalent to the original primitive variable Navier–Stokes problem (1)–(7) in two dimensions provided that the two conditions (14) on the data are satisfied.

Proof. The implication is evident. Conversely, let us assume that ζ and ψ are solutions to the set of equations and conditions (13) with the data $\mathbf{u}_0$ and a satisfying (14). Let us consider for $t > 0$ the vector field $\mathbf{v} = \nabla\psi \times \mathbf{k}$. First, $\mathbf{v}$ is solenoidal, since $\nabla \cdot \mathbf{v} = \nabla \cdot (\nabla\psi \times \mathbf{k}) = 0$. Furthermore, its curl satisfies $\nabla \times \mathbf{v} = \nabla \times (\nabla\psi \times \mathbf{k}) = -\nabla^2\psi\mathbf{k} = \zeta\mathbf{k}$, since $-\nabla^2\psi = \zeta$. Hence the vorticity equation in (13) gives

$$\frac{\partial \nabla \times \mathbf{v}}{\partial t} - \nu \nabla^2 \nabla \times \mathbf{v} + J(\nabla \times \mathbf{v}, \psi) = g\mathbf{k}.$$

By virtue of the vector identity used before and since $\nabla \cdot \mathbf{v} = 0$, the non-linear term can be expressed as

$$\begin{aligned} J(\nabla \times \mathbf{v}, \psi) &= (\mathbf{v} \cdot \nabla)\nabla \times \mathbf{v} = \nabla \times [(\nabla \times \mathbf{v}) \times \mathbf{v}] \\ &= \nabla \times [(\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla(\tfrac{1}{2}v^2)] = \nabla \times [(\mathbf{v} \cdot \nabla)\mathbf{v}]. \end{aligned}$$

Recalling that $g\mathbf{k} = \nabla \times \mathbf{f}$, the vorticity equation gives

$$\nabla \times \left(\frac{\partial \mathbf{v}}{\partial t} - \nu \nabla^2 \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \mathbf{f} \right) = 0$$

and therefore

$$\frac{\partial \mathbf{v}}{\partial t} - \nu \nabla^2 \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \mathbf{f} = \nabla q$$

for some scalar function q. In order that $\mathbf{v}$ can be identified with the velocity field $\mathbf{u}$ solution of the original problem (and q with $-p$), it remains to show that $\mathbf{v}$ assumes the same boundary and initial values of $\mathbf{u}$. For the boundary values it results that $\mathbf{v}|_{\partial\Omega} = \nabla\psi \times \mathbf{k}|_{\partial\Omega}$, which, after separating the normal and tangential components, gives

$$\mathbf{n} \cdot \mathbf{v}|_{\partial\Omega} = \mathbf{n} \cdot \nabla\psi \times \mathbf{k}|_{\partial\Omega} = \mathbf{k} \times \mathbf{n} \cdot \nabla\psi|_{\partial\Omega} = \boldsymbol{\tau} \cdot \nabla\psi|_{\partial\Omega} = \left.\frac{\partial \psi}{\partial s}\right|_{\partial\Omega} = \frac{\partial a}{\partial s} = \mathbf{n} \cdot \mathbf{b},$$

$$\boldsymbol{\tau} \cdot \mathbf{v}|_{\partial\Omega} = \boldsymbol{\tau} \cdot \nabla\psi \times \mathbf{k}|_{\partial\Omega} = \mathbf{k} \times \boldsymbol{\tau} \cdot \nabla\psi|_{\partial\Omega} = -\mathbf{n} \cdot \nabla\psi|_{\partial\Omega} = -\left.\frac{\partial \psi}{\partial n}\right|_{\partial\Omega} = -b = \boldsymbol{\tau} \cdot \mathbf{b}.$$

Hence $\mathbf{v}|_{\partial\Omega} = \mathbf{b} = \mathbf{u}|_{\partial\Omega}$. Concerning the initial values, one has to determine the values assumed by $\mathbf{v}$ as $t \to 0^+$ as a consequence of the imposition of the initial condition for ζ in the ζ–ψ system. One has

$$\mathbf{v}|_{t=0} = \lim_{t \to 0^+} \mathbf{v}(\mathbf{x}, t) = \lim_{t \to 0^+} \nabla\psi(\mathbf{x}, t) \times \mathbf{k} = \nabla\left(\lim_{t \to 0^+} \psi(\mathbf{x}, t)\right) \times \mathbf{k}.$$

Let ψ_0 denote the solution of the Dirichlet problem

$$-\nabla^2\psi_0 = \zeta_0 = \nabla \times \mathbf{u}_0 \cdot \mathbf{k}, \qquad \psi_0|_{\partial\Omega} = a(s, 0).$$

By the assumed continuity of $a(s, t)$ as $t \to 0^+$ and since $\zeta_0 = \zeta|_{t=0}$, the well-posedness of the Dirichlet problem implies that $\lim_{t \to 0^+} \psi(\mathbf{x}, t) = \psi_0(\mathbf{x})$, so that

$$\mathbf{v}|_{t=0} = \nabla\psi_0(\mathbf{x}) \times \mathbf{k}.$$

Using the identity $\nabla \times (\nabla f \times \mathbf{k}) = -\nabla^2 f \mathbf{k}$ in the Poisson equation above, one obtains $\nabla \times (\nabla\psi_0 \times \mathbf{k}) = \nabla \times \mathbf{u}_0$ or $\nabla \times (\nabla\psi_0 \times \mathbf{k} - \mathbf{u}_0) = 0$. It follows that

$$\nabla\psi_0 \times \mathbf{k} - \mathbf{u}_0 = \nabla\alpha$$

for some scalar function α. Now $\nabla \cdot (\nabla\psi_0 \times \mathbf{k}) = 0$ and $\nabla \cdot \mathbf{u}_0 = 0$ by the first condition in (14), so that α is harmonic in Ω. Furthermore, taking the normal component of $\nabla\psi_0 \times \mathbf{k} - \mathbf{u}_0$ on $\partial\Omega$,

$$\mathbf{n} \cdot \nabla\psi_0 \times \mathbf{k}|_{\partial\Omega} - \mathbf{n} \cdot \mathbf{u}_0|_{\partial\Omega} = \mathbf{k} \times \mathbf{n} \cdot \nabla\psi_0|_{\partial\Omega} - \mathbf{n} \cdot \mathbf{u}_0|_{\partial\Omega} = \boldsymbol{\tau} \cdot \nabla\psi_0|_{\partial\Omega} - \mathbf{n} \cdot \mathbf{u}_0|_{\partial\Omega}$$

$$= \left.\frac{\partial \psi_0}{\partial s}\right|_{\partial\Omega} - \mathbf{n} \cdot \mathbf{u}_0|_{\partial\Omega} = \frac{\partial a(s, 0)}{\partial s} - \mathbf{n} \cdot \mathbf{u}_0|_{\partial\Omega} = 0$$

by virtue of the second condition in (14). It follows that $\mathbf{n} \cdot \nabla\alpha|_{\partial\Omega} = \partial\alpha/\partial n|_{\partial\Omega} = 0$, so that $\alpha = \text{constant}$. In conclusion, $\nabla\psi_0 \times \mathbf{k} = \mathbf{u}_0$ everywhere in Ω and therefore $\mathbf{v}|_{t=0} = \mathbf{u}_0$. This completes the proof. □

Remark 1

The vorticity–streamfunction problem could also be stated by specifying the initial condition for the vorticity directly in terms of a prescribed initial vorticity field ζ_0, namely

$$\zeta|_{t=0} = \zeta_0,$$

where ζ_0 is an arbitrary function, with no reference to an initial velocity field. It will be shown further that ζ and ψ are continuous functions with respect to the time variable with values in $H^{-1}(\Omega)$ and $H^1(\Omega)$ respectively. As a result, the Dirichlet problem

$$-\nabla^2\psi(\cdot, t=0) = \zeta_0, \qquad \psi(\cdot, t=0)|_{\partial\Omega} = a(s, 0)$$

is meaningful in the usual weak sense (i.e. in $H^{-1}(\Omega) \times H^{1/2}(\partial\Omega)$). If we set $\tilde{\mathbf{u}}_0 = \nabla\psi(\cdot, t=0) \times \mathbf{k}$, we obtain

$$\zeta_0 = \nabla \times \tilde{\mathbf{u}}_0 \cdot \mathbf{k},$$

we necessarily have $\nabla \cdot \tilde{\mathbf{u}}_0 = 0$ and by classical arguments (Reference 9, p. 27) the boundary condition $\mathbf{n} \cdot \tilde{\mathbf{u}}_0|_{\partial\Omega} = \partial a(s, 0)/\partial s$ is meaningful in some weak sense (in $H^{-1/2}(\partial\Omega)$ as a matter of fact). That is, $\tilde{\mathbf{u}}_0$ necessarily satisfies conditions (14). In conclusion, if continuity of ψ and ζ with respect to time and values in $H^1(\Omega)$ and $H^{-1}(\Omega)$ is assumed, ζ_0 is necessarily the curl of a velocity field $\tilde{\mathbf{u}}_0$ which satisfies (14). As a result, equivalence of the ζ–ψ formulation with some $\mathbf{u}$–p formulation can be achieved only if the initial data of the $\mathbf{u}$–p problem satisfy (14).

Remark 2

If no regularity higher than that of $H^{-1}(\Omega)$ is wanted for ζ_0, the Neumann condition $\partial\psi(\cdot, t=0)/\partial n|_{\partial\Omega} = b(s, 0)$ is meaningless; hence no compatibility condition is required for b at $t = 0$. However, if $\zeta_0 \in L^2(\Omega)$, the normal derivative of $\psi(\cdot, t=0)$ is meaningful in $H^{-1/2}(\partial\Omega)$ (Reference 9, p. 27) and $b(s, t=0) = \partial\psi(\cdot, t=0)/\partial n|_{\partial\Omega}$ is a compatibility condition that must be satisfied by b. If b does not satisfy it, there is no possibility for ψ to be in $C^0([0, T]; H^2(\Omega))$; in other words, $\|\psi\|_2$ will necessarily blow up as $t \to 0$. This loss of regularity as $t \to 0$ is similar to that already discovered for the $\mathbf{u}$–p formulation in the previous section. Indeed, the compatibility condition in question amounts to

$$\mathbf{n} \times \mathbf{b}|_{t=0} = \mathbf{n} \times \mathbf{u}_0|_{\partial\Omega} \tag{15}$$

and this condition has been shown to be violated in the case of an impulsive start.

Now, in order to prove the equivalence of the ζ–ψ and $\mathbf{u}$–p formulations in some rigorous way, we introduce some definitions and preliminary results.

4. DEFINITIONS AND PRELIMINARIES

In the following, the set of real functions infinitely differentiable and of compact support in Ω is denoted by $\mathscr{D}(\Omega)$. The set of distributions on Ω is denoted by $\mathscr{D}'(\Omega)$. Spaces of vector-valued functions are hereafter denoted with boldface type, though no distinction is made in the notation of inner products and norms.

In order to have some unitary framework for the curl operator in a two-dimensional space, we introduce

$$\mathbf{curl}: \quad \mathscr{D}'(\Omega) \to \boldsymbol{\mathscr{D}}'(\Omega)$$

$$\phi \mapsto \left(\frac{\partial\phi}{\partial y}, -\frac{\partial\phi}{\partial x}\right) \tag{16}$$

$$\mathrm{curl}: \quad \boldsymbol{\mathscr{D}}'(\Omega) \to \mathscr{D}'(\Omega)$$

$$\mathbf{v} \mapsto \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \tag{17}$$

Note that we have $\mathbf{curl}\,\phi = \nabla\phi \times \mathbf{k}$ and $\mathrm{curl}\,\mathbf{v} = \nabla \times \mathbf{v}\cdot\mathbf{k}$. Note also that **curl** and curl are transpose to each other in the following sense.

$$\forall \mathbf{v} \in \mathscr{D}'(\Omega),\ \forall \phi \in \mathscr{D}(\Omega), \quad \langle \mathrm{curl}\,\mathbf{v}, \phi\rangle = \langle \mathbf{v}, \mathbf{curl}\,\phi\rangle, \tag{18}$$

$$\forall \mathbf{v} \in \mathscr{D}(\Omega),\ \forall \phi \in \mathscr{D}'(\Omega), \quad \langle \phi, \mathrm{curl}\,\mathbf{v}\rangle = \langle \mathbf{curl}\,\phi, \mathbf{v}\rangle. \tag{19}$$

It is a simple matter of calculus to show the following identities which will be used repeatedly:

$$\forall \mathbf{v} \in \mathscr{D}'(\Omega), \quad \mathbf{curl}\,\mathrm{curl}\,\mathbf{v} = -\nabla^2\mathbf{v} + \nabla(\nabla\cdot\mathbf{v}), \tag{20}$$

$$\forall \phi \in \mathscr{D}'(\Omega), \quad \mathrm{curl}\,\mathbf{curl}\,\phi = -\nabla^2\phi. \tag{21}$$

As usual, $L^2(\Omega)$ denotes the space of real-valued functions, the squares of which are summable in Ω. We denote the inner product in $L^2(\Omega)$ by $(\cdot,\cdot)$ and let $\|\cdot\|_0$ be its norm. $H^m(\Omega)$, $m \geqslant 0$, is the set of distributions the successive derivatives of which, up to order m, can be identified with square summable functions. The space $H^m(\Omega)$, equipped with the norm

$$\|u\|_m = \left(\sum_{|\alpha|=0}^{m} \|D^\alpha u\|_0^2\right)^{1/2},$$

expressed in the multi-index notation, is a Hilbert space. Now we define $H_0^m(\Omega)$ as the completion of $\mathscr{D}(\Omega)$ in $H^m(\Omega)$ and we denote $H^{-m}(\Omega)$ the dual of $H_0^m(\Omega)$. The duality product is denoted by $(\cdot,\cdot)$.

Analogues to (18) and (19) are now given by the following.

Lemma 1

Let $m \geqslant 0$:

$$\forall \mathbf{f} \in \mathbf{H}^{-m}(\Omega),\ \forall \phi \in H_0^{m+1}(\Omega), \quad (\mathrm{curl}\,\mathbf{f}, \phi) = (\mathbf{f}, \mathbf{curl}\,\phi), \tag{22}$$

$$\forall \mathbf{f} \in \mathbf{H}_0^{m+1}(\Omega),\ \forall \phi \in H^{-m}(\Omega), \quad (\phi, \mathrm{curl}\,\mathbf{f}) = (\mathbf{curl}\,\phi, \mathbf{f}). \tag{23}$$

Proof. Use density of $\mathscr{D}(\Omega)$ in $H_0^{m+1}(\Omega)$ and continuity of duality product. □

The analysis of the Navier–Stokes equations leads us to consider solenoidal velocity fields; hence we define $\mathscr{J}(\Omega) = \{\mathbf{v} \in \mathscr{D}(\Omega),\ \nabla\cdot\mathbf{v} = 0\}$ and we denote $\mathbf{J}_0^m(\Omega)$, $m \geqslant 0$, the completion of $\mathscr{J}(\Omega)$ in $\mathbf{H}^m(\Omega)$. Spaces $\mathbf{J}_0^0(\Omega)$ and $\mathbf{J}_0^1(\Omega)$ are characterized by the following.

Theorem 2

If Ω is open, bounded and Lipschitz in the plane, then

$$\mathbf{J}_0^0(\Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega);\ \nabla\cdot\mathbf{v} = 0,\ \mathbf{n}\cdot\mathbf{v}|_{\partial\Omega} = 0\},$$

$$\mathbf{J}_0^1(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega);\ \nabla\cdot\mathbf{v} = 0,\ \mathbf{v}|_{\partial\Omega} = 0\}.$$

Proof. See e.g. Reference 2 (pp. 15–18). □

An important connection of the spaces above with the curl and **curl** operators is given by the following.

Lemma 2

Assuming Ω is simply connected, then we have the following isomorphisms:

$$\begin{aligned}
&\text{(i) } \mathbf{curl}: && H_0^1(\Omega) \to \mathbf{J}_0^0(\Omega),\\
&\text{(ii) } \mathbf{curl}: && H_0^2(\Omega) \to \mathbf{J}_0^1(\Omega),\\
&\text{(iii) curl}: && \mathbf{J}_0^0(\Omega) \to H^{-1}(\Omega),\\
&\text{(iv) curl}: && \nabla^2\mathbf{J}_0^1(\Omega) \to H^{-2}(\Omega).
\end{aligned}$$

Proof. (i) The range of **curl** is in $\mathbf{J}_0^0(\Omega)$. Let $\phi \in H_0^1(\Omega)$ and define $\{\phi_n\} \in \mathscr{D}(\Omega)^{\mathbb{N}}$ a sequence so that $\phi_n \to \phi$ in $H_0^1(\Omega)$. It is easy to see that the sequence $\{\mathbf{curl}\ \phi_n\}$ is in $\mathscr{J}(\Omega)$ and that $\mathbf{curl}\ \phi_n \to \mathbf{curl}\ \phi$ in $\mathbf{L}^2(\Omega)$; as a result $\mathbf{curl}\ \phi$ is necessarily in $\mathbf{J}_0^0(\Omega)$ by definition of $\mathbf{J}_0^0(\Omega)$.
Continuity. The continuity is evident: $\|\mathbf{curl}\ \phi\|_0 \leqslant \|\phi\|_1$.
Injectivity. Let $\phi \in H_0^1(\Omega)$. Then

$$\begin{aligned}
\mathbf{curl}\ \phi = 0 &\Rightarrow \text{curl } \mathbf{curl}\ \phi = 0\\
&\Rightarrow \nabla^2\phi = 0 \quad \text{by} \quad (21)\\
&\Rightarrow \phi = 0 \quad \text{since } \phi|_{\partial\Omega} = 0.
\end{aligned}$$

Surjectivity. Let $\mathbf{f} \in \mathbf{J}_0^0(\Omega)$; there is $\phi \in H_0^1(\Omega)$ such that $-\nabla^2\phi = \text{curl }\mathbf{f}$, i.e. $\text{curl}\,(\mathbf{curl}\ \phi - \mathbf{f}) = 0$. As a result of Lemma 3 below which generalizes Stokes' theorem, there is $p \in H^1(\Omega)/\mathbb{R}$ so that $\mathbf{curl}\ \phi - \mathbf{f} = \nabla p$. Hence $\nabla p \in \mathbf{J}_0^0(\Omega)$; in other words $\nabla^2 p = 0$ and $\partial p/\partial n|_{\partial\Omega} = 0$; as a consequence $p = \text{constant}$ and $\mathbf{curl}\ \phi = \mathbf{f}$.

(ii) The range of **curl** is in $\mathbf{J}_0^1(\Omega)$; continuity and injectivity as in (i).
Surjectivity. Let $\mathbf{f} \in \mathbf{J}_0^1(\Omega)$; from (i) we infer that there is $\phi \in H_0^1(\Omega)$ such that $\mathbf{curl}\ \phi = (\partial\phi/\partial y, -\partial\phi/\partial x) = (f_x, f_y) = \mathbf{f}$. Since $\mathbf{f} \in \mathbf{J}_0^1(\Omega) \subset \mathbf{H}^1(\Omega)$, it follows that $\partial\phi/\partial x \in H^1(\Omega)$ and $\partial\phi/\partial y \in H^1(\Omega)$, i.e. $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$. Furthermore, since $\mathbf{f} \in \mathbf{H}_0^1(\Omega)$, we have $\nabla\phi|_{\partial\Omega} = 0$; hence $\partial\phi/\partial n|_{\partial\Omega} = 0$, which means $\phi \in H_0^2(\Omega)$.

(iii) Is obtained from (i) by duality, since $\mathbf{J}_0^0(\Omega)' \equiv \mathbf{J}_0^0(\Omega)$.

(iv) Results from (ii). Let $f \in H^{-2}(\Omega)$. Consider the problem: find $\mathbf{u} \in \mathbf{J}_0^1(\Omega)$ such that $\text{curl } \nabla^2\mathbf{u} = f$ in $H^{-2}(\Omega)$. Since $H_0^2(\Omega) = \mathbf{curl}^{-1}\ \mathbf{J}_0^1(\Omega)$, this problem is equivalent to: find $\mathbf{u} \in \mathbf{J}_0^1(\Omega)$ such that $-(\nabla\mathbf{u}, \nabla\mathbf{v}) = (f, \mathbf{curl}^{-1}\ \mathbf{v})$ for all $\mathbf{v} \in \mathbf{J}_0^1(\Omega)$. Such a problem is well-posed. □

We now state the main result, which involves the simple connectivity of the domain Ω.

Lemma 3

If Ω is simply connected, for $\mathbf{u} \in \mathbf{L}^2(\Omega)$ we have

$$(\text{curl } \mathbf{u} = 0 \quad \text{in} \quad \Omega) \quad \Leftrightarrow \quad (\exists!\, p \in H^1(\Omega)/\mathbb{R};\ \mathbf{u} = \nabla p).$$

Proof. See e.g. Reference 9 (p. 31). □

It is necessary for later results to isolate solenoidal distributions of $\mathbf{H}^{-1}(\Omega)$ and solenoidal velocity fields in $\mathbf{L}^2(\Omega)$. This is achieved by the following result.

Lemma 4

We have the decomposition

$$\mathbf{H}^{-1}(\Omega) = \nabla^2\mathbf{J}_0^1(\Omega) \oplus \nabla(L^2(\Omega)/\mathbb{R}).$$

Proof. This is equivalent to the fact that Stokes' problem is well-posed; see e.g. Reference 2 (pp. 21–24). □

Note that the distributions of $\nabla^2 \mathbf{J}_0^1(\Omega)$ are solenoidal and $\nabla^2 \mathbf{J}_0^1(\Omega)$ is isomorphic with the dual of $\mathbf{J}_0^1(\Omega)$. Since $\nabla^2 \mathbf{J}_0^1(\Omega)$ is a closed subspace of $\mathbf{H}^{-1}(\Omega)$, we can define the orthogonal projection on it; such a projection is hereafter denoted by Π_{-1}: $\mathbf{H}^{-1}(\Omega) \to \nabla^2 \mathbf{J}_0^1(\Omega)$.

Lemma 5

We have the decomposition

$$\mathbf{L}^2(\Omega) = \mathbf{J}_0^0(\Omega) \oplus \nabla(H^1(\Omega)/\mathbb{R}).$$

Proof. Let $\mathbf{f} \in \mathbf{L}^2(\Omega)$; there is a unique $p \in H^1(\Omega)/\mathbb{R}$ such that

$$\forall \varphi \in H^1(\Omega), \quad (\nabla p, \nabla \varphi) = (\mathbf{f}, \nabla \varphi).$$

Then set $\mathbf{u} = \mathbf{f} - \nabla p$. One verifies easily that since $\nabla \cdot \mathbf{f} = \nabla^2 p$ in $H^{-1}(\Omega)$, $\mathbf{u} \in \mathbf{J}_0^0(\Omega)$. □

Note that $\mathbf{J}_0^0(\Omega)$ is closed in $\mathbf{L}^2(\Omega)$ so that one can define the orthogonal projection on it; it is hereafter denoted by Π_0: $\mathbf{L}^2(\Omega) \to \mathbf{J}_0^0(\Omega)$.

Finally, we state the result which, combined with Lemma 2, will enable us to prove the equivalence of the $\mathbf{u}$–p formulation of the Navier–Stokes equations with the ζ–ψ one.

Lemma 6

Let E and F be two Banach spaces and $\mathscr{A}: E \to F$ be an isomorphism. Let $T > 0$ and $1 \leqslant p \leqslant \infty$ and define

$$\mathscr{A}_p: \quad L^p((0,T);E) \to L^p((0,T);F)$$
$$\mathscr{A}_c: \quad C([0,T];E) \to C([0,T];F)$$
$$w(t) \mapsto \mathscr{A}(w(t))$$

Then $\mathscr{A}_p$ and $\mathscr{A}_c$ are isomorphisms.

5. THE u–p FORMULATION

In this section we recall classical results concerning the $\mathbf{u}$–p formulation that will be used to show the well-posedness of the ζ–ψ problem. Homogeneous boundary conditions are assumed for the sake of simplicity. Let $T > 0$ and consider the unsteady Navier–Stokes equations on the time interval $(0, T)$. One classical way of looking at this problem is as follows (Reference 10, pp. 64–78):

$$\mathscr{P}_1\{\mathbf{f}, \mathbf{u}_0\} \begin{cases} \text{for } \mathbf{f} \in L^2((0,T);\mathbf{H}^{-1}(\Omega)) \text{ and } \mathbf{u}_0 \in \mathbf{J}_0^0(\Omega) \text{ find} \\ \mathbf{u} \in L^2((0,T);\mathbf{J}_0^1(\Omega)) \cap L^\infty((0,T);\mathbf{J}_0^0(\Omega)) \text{ and} \\ p \in \mathscr{D}'(]0,T[\times\Omega) \text{ such that} \\ \dfrac{\partial \mathbf{u}}{\partial t} - \nu\nabla^2\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \mathscr{D}'(]0,T[\times\Omega), \\ \mathbf{u}|_{t=0} = \mathbf{u}_0. \end{cases}$$

Note that this problem is not of Cauchy–Kowalewsky type, i.e. there is no dynamical equation for the pressure. The main difficulty associated with this fact, as noted in Section 3, is that when coming to a numerical approximation to $\mathscr{P}_1\{\mathbf{f}, \mathbf{u}_0\}$, the pressure field cannot be calculated by an explicit time advancement procedure. One possibility for circumventing this major difficulty is to take the quotient of the dynamical equation by all the gradients or, equivalently, project it on to $\nabla^2 \mathbf{J}_0^1(\Omega)$. Thus we consider the problem

$$\mathscr{P}_2\{\mathbf{f}, \mathbf{u}_0\}\begin{cases}\text{for } \mathbf{f} \in L^2((0, T); \mathbf{H}^{-1}(\Omega)) \text{ and } \mathbf{u}_0 \in \mathbf{J}_0^0(\Omega) \text{ find}\\ \mathbf{u} \in L^2((0, T); \mathbf{J}_0^1(\Omega)) \cap L^\infty((0, T); \mathbf{J}_0^0(\Omega)) \text{ such that}\\ \left(\dfrac{\partial \mathbf{u}}{\partial t}, \mathbf{v}\right) + \nu a(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = (\Pi_{-1}\,\mathbf{f}, \mathbf{v}) \quad \text{in } \mathscr{D}'(]0, T[),\\ \forall \mathbf{v} \in \mathbf{J}_0^1(\Omega) \cap \mathbf{L}^n(\Omega),\\ \mathbf{u}|_{t=0} = \mathbf{u}_0.\end{cases}$$

where n is the dimension of the physical space (hereafter $n = 2$) and the forms a and b are defined by

$$a(\mathbf{u}, \mathbf{v}) = (\nabla \mathbf{u}, \nabla \mathbf{v}), \qquad b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = ((\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w}).$$

Remark 3

The trilinear form b: $\mathbf{J}_0^1(\Omega) \times \mathbf{J}_0^1(\Omega) \times \mathbf{L}^n(\Omega) \to \mathbb{R}$ is continuous, but for space dimension $n \leqslant 4$ we have $\mathbf{J}_0^1(\Omega) \cap \mathbf{L}^n(\Omega) = \mathbf{J}_0^1(\Omega)$, so we no longer bother with $\mathbf{L}^n(\Omega)$ (Reference 10, p. 66).

The important point is the following.

Theorem 3

Problems $\mathscr{P}_1\{\mathbf{f}, \mathbf{u}_0\}$ and $\mathscr{P}_2\{\mathbf{f}, \mathbf{u}_0\}$ are equivalent.

Proof. (a) $\mathscr{P}_1 \Rightarrow \mathscr{P}_2$. Let $\mathbf{u}$ be a solution to $\mathscr{P}_1\{\mathbf{f}, \mathbf{u}_0\}$. Then $\forall \boldsymbol{\phi} \in \mathscr{J}(\Omega)$,

$$\left(\frac{\partial \mathbf{u}}{\partial t}, \boldsymbol{\phi}\right) + \nu a(\mathbf{u}, \boldsymbol{\phi}) + b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}) = (\mathbf{f}, \boldsymbol{\phi}) = (\Pi_{-1}\,\mathbf{f}, \boldsymbol{\phi}) \quad \text{in } \mathscr{D}'(]0, T[).$$

$\mathscr{J}(\Omega)$ being dense in $\mathbf{J}_0^1(\Omega) \cap \mathbf{L}^n(\Omega)$, we can take the limit, i.e. $\forall \mathbf{v} \in \mathbf{J}_0^1(\Omega) \cap \mathbf{L}^n(\Omega)$,

$$\left(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v}\right) + \nu a(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = (\Pi_{-1}\,\mathbf{f}, \mathbf{v}) \quad \text{in } \mathscr{D}'(]0, T[).$$

Hence $\mathbf{u}$ is a solution to $\mathscr{P}_2\{\mathbf{f}, \mathbf{u}_0\}$.

(b) $\mathscr{P}_2 \Rightarrow \mathscr{P}_1$. Let $\mathbf{u}$ be a solution to $\mathscr{P}_2\{\mathbf{f}, \mathbf{u}_0\}$. Then the linear form

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \nabla^2 \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \Pi_{-1}\,\mathbf{f}$$

vanishes on $\mathscr{D}(]0, T[; \mathbf{J}_0^1(\Omega))$. Furthermore, since there is some $\tilde{p} \in L^2(\Omega)/\mathbb{R}$ such that $\mathbf{f} = \Pi_{-1}\,\mathbf{f} + \nabla \tilde{p}$ according to Lemma 4, the conclusion above holds also for the linear form

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \nabla^2 \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \Pi_{-1}\,\mathbf{f} - \nabla \tilde{p}.$$

As a result of De Rham's theorem,[11] there is $p \in \mathscr{D}'(]0, T[\times \Omega)$ such that

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\nabla^2\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mathbf{f} = -\nabla p \quad \text{in } \mathscr{D}'(]0, T[\times \Omega).$$

Hence $(\mathbf{u}, p)$ is a solution to $\mathscr{P}_1\{\mathbf{f}, \mathbf{u}_0\}$.

Remark 4

For arbitrary n and finite time T it can be shown that $\mathscr{P}_2\{\mathbf{f}, \mathbf{u}_0\}$ has at least one solution. Furthermore, the initial condition may be given some precise sense (Reference 10, pp. 64–106). It turns out that in 2D ($n = 2$) there is uniqueness of the solution and the velocity field belongs to $C([0, T]; \mathbf{J}^0_0(\Omega))$. In other words, in 2D, $\mathbf{u}(t)$ with values in $\mathbf{J}^0_0(\Omega)$ is a continuous function with respect to the time variable. In this context the initial condition is easily interpreted.

We turn now to the 2D formulation in terms of the vorticity and streamfunction, always assuming that Ω is simply connected.

6. THE ζ–ψ FORMULATION

In this section we consider the problem

$$\mathscr{P}_3\{g, \zeta_0\}\begin{cases} \text{for } g \in L^2((0, T); H^{-2}(\Omega)) \text{ and } \zeta_0 \in H^{-1}(\Omega) \text{ find} \\ \zeta \in L^2((0, T); L^2(\Omega)) \cap C([0, T]; H^{-1}(\Omega)) \text{ and} \\ \psi \in L^2((0, T); H^2_0(\Omega)) \cap C([0, T]; H^1_0(\Omega)) \text{ such that} \\ \dfrac{\partial \zeta}{\partial t} - \nu\nabla^2\zeta + \mathbf{curl}\,(\psi)\cdot\nabla\zeta = g \quad \text{in } \mathscr{D}'(]0, T[; H^{-2}(\Omega)), \\ -\nabla^2\psi = \zeta \quad \text{in } L^2(\Omega), \\ \zeta|_{t=0} = \zeta_0 \quad \text{in } H^{-1}(\Omega). \end{cases}$$

One question that arises at this point is whether g and ζ_0 are the curls of some body force and initial velocity field respectively. The answer to this question is given by the following.

Lemma 7

(i) There is some $\mathbf{f}$ in $L^2((0, T); \mathbf{H}^{-1}(\Omega))$ and a *unique* $\Pi_{-1}\mathbf{f}$ in $L^2((0, T); \nabla^2\mathbf{J}^1_0(\Omega))$ such that $g = \operatorname{curl}(\Pi_{-1}\mathbf{f}) = \operatorname{curl}(\mathbf{f})$.

(ii) There is some $\mathbf{u}_0$ in $L^2((0, T); \mathbf{L}^2(\Omega))$ and a *unique* $\Pi_0\mathbf{u}_0$ in $L^2((0, T); \mathbf{J}^0_0(\Omega))$ such that $\zeta_0 = \operatorname{curl}(\Pi_0\mathbf{u}_0) = \operatorname{curl}(\mathbf{u}_0)$.

Proof. Apply parts (ii) and (iv) of Lemma 2 together with Lemmas 4 and 5.

The other question that has to be addressed now concerns the equivalence of $\mathscr{P}_3\{g, \zeta_0\}$ with the same Navier–Stokes problem of type $\mathscr{P}_2\{\mathbf{f}, \mathbf{u}_0\}$. It turns out, as shown below, that $\mathscr{P}_3\{g, \zeta_0\}$ is equivalent to $\mathscr{P}_2\{\Pi_{-1}\mathbf{f}, \Pi_0\mathbf{u}_0\}$. Indeed, the only way to impose that $\mathscr{P}_3\{g, \zeta_0\}$ be equivalent to a unique Navier–Stokes problem of type $\mathscr{P}_2$ is to require the initial velocity field of $\mathscr{P}_2$ to be in $\mathbf{J}^0_0(\Omega)$. In other words, the equivalence of a ζ–ψ problem with some definite $\mathbf{u}$–p problem is achievable only if the initial velocity field of the $\mathbf{u}$–p problem is in $\mathbf{J}^0_0(\Omega)$, namely

$$\nabla\cdot\mathbf{u}_0 = 0 \quad \text{and} \quad \mathbf{n}\cdot\mathbf{u}_0|_{t=0} = 0.$$

These conditions coincide with those considered in (14) in the particular case of homogeneous boundary conditions for the velocity.

Theorem 4

Problem $\mathscr{P}_3\{g, \zeta_0\}$ is equivalent to $\mathscr{P}_2\{\Pi_{-1}\mathbf{f}, \Pi_0\mathbf{u}_0\}$.

Proof. (a) Note that in 2D the solution velocity field $\mathbf{u}$ is in $C([0,T]; \mathbf{J}_0^0(\Omega))$ and $C([0,T]; \mathbf{J}_0^0(\Omega)) \subset L^\infty((0,T); \mathbf{J}_0^0(\Omega))$. As a result, $\mathscr{P}_2\{\mathbf{f}, \mathbf{u}_0\}$ is still well-posed if we restrict the solution to be in $L^2((0,T); \mathbf{J}_0^1(\Omega)) \cap C([0,T]; \mathbf{J}_0^0(\Omega))$.

(b) According to Lemmas 2 and 6, assuming that *there is a unique* $\mathbf{u}$ solution to $\mathscr{P}_2\{\Pi_{-1}\mathbf{f}, \Pi_0\mathbf{u}_0\}$ is equivalent to assuming that *there is a unique*

$$\psi \in L^2((0,T); H_0^2(\Omega)) \cap C([0,T]; H_0^1(\Omega))$$

such that $\mathbf{curl}\,\psi = \mathbf{u}$.

(c) Furthermore, denoting $\zeta = -\nabla^2\psi$, we necessarily have

$$\zeta \in L^2((0,T); L^2(\Omega)) \cap C([0,T]; H^{-1}(\Omega)).$$

(d) According to part (i) of Lemma 2, the momentum equation of $\mathscr{P}_2\{\mathbf{f}, \mathbf{u}_0\}$ is equivalent to

$$\left(\frac{\partial}{\partial t}\mathbf{curl}\,\psi, \mathbf{curl}\,\phi\right) + \nu a(\mathbf{curl}\,\psi, \mathbf{curl}\,\phi) + b(\mathbf{curl}\,\psi, \mathbf{curl}\,\psi, \mathbf{curl}\,\phi) = (\Pi_{-1}\mathbf{f}, \mathbf{curl}\,\phi) \quad \text{in } \mathscr{D}'(]0,T[),$$

$\forall\phi \in H_0^2(\Omega)$.

It is decomposed as follows.

(i) $$\forall\gamma \in \mathscr{D}(]0,T[), \quad \left(\left(\frac{\partial}{\partial t}\mathbf{curl}\,\psi, \mathbf{curl}\,\phi\right), \gamma\right) = -\left((\mathbf{curl}\,\psi, \mathbf{curl}\,\phi), \frac{\mathrm{d}\gamma}{\mathrm{d}t}\right).$$

This is a consequence of the fact that $\mathbf{u} \in L^2((0,T); \mathbf{J}_0^1(\Omega)')$ (Reference 10, p. 64) and far-reaching density and trace theorems (Reference 3, pp. 14 and 23; Reference 12, p. 575). Then, thanks to Lemma 1 and (21), we have

$$\forall\gamma \in \mathscr{D}(]0,T[), \quad \left(\left(\frac{\partial}{\partial t}\mathbf{curl}\,\psi, \mathbf{curl}\,\phi\right), \gamma\right) = \left(\left(\frac{\partial\zeta}{\partial t}, \phi\right), \gamma\right).$$

(ii) The diffusion term is easily shown to yield

$$\nu a(\mathbf{curl}\,\psi, \mathbf{curl}\,\phi) = -\nu(\nabla^2\zeta, \phi).$$

(iii) The convection term yields

$$\begin{aligned} b(\mathbf{curl}\,\psi, \mathbf{curl}\,\psi, \mathbf{curl}\,\phi) &= (\mathrm{curl}\,(\mathrm{curl}\,(\mathbf{u})\mathbf{k} \times \mathbf{u} + \nabla(\tfrac{1}{2}u^2)), \phi) \\ &= (\mathbf{u} \cdot \nabla(\mathrm{curl}\,\mathbf{u}), \phi) \\ &= (\mathbf{curl}\,\psi \cdot \nabla(\mathrm{curl}\,\mathbf{curl}\,\psi), \phi) \\ &= (\mathbf{curl}\,(\psi) \cdot \nabla\zeta, \phi). \end{aligned}$$

(iv) The source term gives

$$(\Pi_{-1}\mathbf{f}, \mathbf{curl}\,\phi) = (\mathrm{curl}\,\Pi_{-1}\mathbf{f}, \phi) = (g, \phi).$$

As a result the momentum equation can be *equivalently* written in the form

$$\frac{\partial \zeta}{\partial t} - \nu \nabla^2 \zeta + \mathbf{curl}\,(\psi) \cdot \nabla \zeta = g \quad \text{in } \mathscr{D}'(]0, T[; H^{-2}(\Omega)).$$

(e) It remains to be shown in which terms the initial conditions are equivalent (see Remark 1). Since we want ψ to be in $C([0, T]; H_0^1(\Omega))$, we necessarily have

$$-\nabla^2 \psi(\cdot, t = 0) = \zeta_0 \quad \text{in } H^{-1}(\Omega),$$

i.e. curl ($\mathbf{curl}\, \psi(\cdot, t = 0) - \Pi_0 \mathbf{u}) = 0$ in $H^{-1}(\Omega)$. However, $\mathbf{curl}\, \psi(\cdot, t = 0) - \Pi_0 \mathbf{u}_0 \in \mathbf{J}_0^0(\Omega)$; hence, according to part (iii) of Lemma 2,

$$\mathbf{curl}\, \psi(\cdot, t = 0) = \Pi_0 \mathbf{u}_0.$$

In other words, specifying $\zeta|_{t=0} = \zeta_0$ in problem $\mathscr{P}_3$ is equivalent to specifying $\mathbf{u}|_{t=0} = \Pi_0 \mathbf{u}_0$ in problem $\mathscr{P}_2$. □

Remark 5

The key point here is that we require $\psi \in C([0, T]; H_0^1(\Omega))$, which in terms of velocity (according to Lemma 6) is equivalent to requiring $\mathbf{u}$ to be in $C([0, T]; \mathbf{J}_0^0(\Omega))$. In other words, ψ cannot be in $C([0, T]; H_0^1(\Omega))$ unless $\mathbf{u}(\cdot, t = 0)$ is in $\mathbf{J}_0^0(\Omega)$, i.e. satisfies (14).

Remark 6

Problem $\mathscr{P}_3\{g, \zeta_0\}$ is well-posed.

7. CONCLUSIONS

Equivalence of the vorticity–streamfunction equations with the primitive variable formulation of the unsteady incompressible Navier–Stokes equations has been established. One formal proof and a mathematical proof with a rigorous setting have been given so that the arguments are intelligible to as wide an audience as possible. It has been shown that minimal compatibility conditions on the initial data and the boundary condition are required so that existence and unicity of a solution of minimal (reasonable) regularity are ensured. These compatibility conditions have been shown to play an important role in establishing the equivalence of the two formulations referred to above. An additional compatibility condition concerning the tangential components of the initial velocity field and the boundary condition is to be satisfied if higher regularity of the solution is needed as $t \to 0$. Anyhow, in the case of an impulsive start, the H^1-norm of the velocity and the H^2-norm of the streamfunction necessarily blow up as $t \to 0$.

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